Point-weight designs with design conditions on t points

Alexander W. Dent

Mathematics Dept., Royal Holloway, University of London, Egham Hill, Egham, Surrey, TW20 0EX, United Kingdom

Abstract

This paper examines some of the properties of point-weight incidence structures, i.e. incidence structures for which every point is assigned a positive integer weight. In particular it examines point-weight designs with a design condition that stipulates that any two "identical" sets of t points must lie on the same number of blocks. We introduce a new class of designs with this property: row-sum designs, and examine the basic properties of row-sum point-weight designs and their similarities to classical (non-point-weight) designs and the point-weight designs of Horne (1996).

Key words: block design, point-weight design, row-sum design, point-sum design, n-ary block design

1 Introduction

Design theory, in its purest form, has existed for hundreds of years and in that time there have been many interesting generalisations and new formulations, driven either by academic curiosity or practical need. Of these generalisations, it seems that the simplest usually present the most interesting properties. One of the simplest generalisations of classical design theory is point-weight design theory. The concept of a point-weight design was first introduced by Horne [4] and Powlesland [6] in mid-1990s.

A point-weight incidence structure is a structure where every point has a positive integer weight associated with it. A point-weight design is a point-weight incidence structure with properties that resemble those properties that define

Email address: alex@fermat.ma.rhul.ac.uk (Alexander W. Dent).
URL: http://www.isg.rhul.ac.uk/~alex/ (Alexander W. Dent).

a classical (non point-weight) design. In particular, a point-weight design in which every point has the same weight is (essentially) a classical design.

This paper intends to to explore the properties of point-weight designs that specify that every set of t points must be incident some set, calculable number of blocks. We will refer to these structures as t-point-weight designs. First we will introduce some existing material on the subject, including some results about point-sum designs [4]. Point-sum designs form the class of point-weight designs that most closely resemble classical designs, and are the most logical extension of them. In the next section we prove some general results about t-point-weight designs and then introduce a new class of t-point-weight design called a *row-sum design*, and prove some basic results about this new class of designs. Row-sum designs share certain similarities with point-sum designs but also have certain quirks of their own. They can also be thought of as a class of n-ary block designs [3, 10]. Lastly we will investigate the relationship between row-sum and point-sum designs. We will conjecture that a t-point-sum design cannot be a t'-row-sum design for any $t' < t$. On the other hand we will show that there do exist point-weight incidence structures that are t-point-sum designs and t'-row-sum designs for $t' \geq t$.

For an introduction to classical design theory, the reader is referred to [1,5,9].

2 Point-Weight Designs

The preliminary ideas of point-weight design theory can be found in [4,6]. We start by defining a point weight incidence structure. A point-weight incidence structure extends a classical incidence structure by assigning every point with a positive integer weight.

Definition 1 (Point-Weight Incidence structure) *A point-weight incidence structure is a quadruple $(V, \mathcal{B}, I, w)$ where V is a set of points, $\mathcal{B}$ is a set of blocks, $I \subseteq V \times \mathcal{B}$ is an incidence relation and $w : V \to \mathbb{Z}^+$ is a function, known as the weight function, that assigns a positive integer "weight" to each point of V.*

We refer to an element $x \in V$ as a 'point', an element $B \in \mathcal{B}$ as a block. If $(x, B) \in I$ then we say that 'x lies on B' or 'B contains x'.

It is prudent at this time to introduce some simple notation. If $(V, \mathcal{B}, I, w)$ is a point-weight incidence structure and $S \subseteq V$ is a set of points then $\sigma(S)$ is the sum of the weights of the points of S and $\iota(S)$ is the number of blocks with which S is incident. In other words:

$$\sigma(S) = \sum_{x \in S} w(x)\,, \tag{2.1}$$

$$\iota(S) = |\{B \in \mathcal{B} : S \subseteq B\}|\,. \tag{2.2}$$

Initially the aim of point-weight design theory is to extend classical design theory in a natural way. In a classical design there are two conditions that an incidence structure has to fulfil in order to be a design: a *constant block size* condition and a *design condition*. In a classical $t-(v,k,\lambda)$ design, the constant block size condition mandates that each block must contain exactly k points and the design condition mandates that every t points must lie on exactly λ blocks. The natural extension of the constant block size condition to point-weight designs is obvious: the sum of the weights of the points on every block must be constant. However, the natural extension of the design condition is less obvious.

The only constraint that seems obvious in choosing a natural candidate for a design condition is that it would be convenient if a design in which all the points have the same weight had the same structure as a classical design. In other words, if $(V,\mathcal{B},I,w)$ is a point-weight design with $w(x) = w(y)$ for all $x, y \in V$ then $(V,\mathcal{B},I)$ is a classical design. This leads nicely to the definition of an underlying structure.

Definition 2 *If* $(V,\mathcal{B},I,w)$ *is a point-weight incidence structure* $\mathcal{S}$ *then* $(V,\mathcal{B},I)$ *is called the* underlying incidence structure $\mathcal{U}$ of $\mathcal{S}$.

Hence, the underlying structure of a point-weight design in which all the points have the same weight should be a classical design. There are two known examples of design conditions with interesting properties.

The first was introduced by Horne [4] and is now termed a *point-sum design*. It is of particular relevance to this paper because the design condition of a point-sum design affects sets of t points.

Definition 3 (Point-sum point-weight design) *A point-weight incidence structure* $(V,\mathcal{B},I,w)$ *is a* $t-(v,k,\lambda;W)$ *point-sum design if*

(1) the sum of the weights of the points is v*, i.e.* $\sigma(V) = v$.
(2) the sum of the weights of the points on any one block is k*, i.e.* $\sigma(B) = k$ *for all* $B \in \mathcal{B}$.
(3) any set $S \subseteq V$ *of* t *points is incident with* λ *blocks, i.e.* $\iota(S) = \lambda$ *for all* $S \subseteq V$ *such that* $|S| = t$.
(4) the image of the weight function is W*, i.e.* $W = Im\, w$.

Several examples and generic methods for constructing a point-sum design are given by Horne [4], including the example of Figure 1.

$$
\left[\begin{array}{cccccccccccc|cccccc}
0&0&0&3&3&3&0&0&0&0&0&0&3&3&3&0&0&0\\
0&0&0&0&0&0&3&3&3&0&0&0&3&0&0&3&3&0\\
3&3&3&0&0&0&0&0&0&0&0&0&0&3&0&3&0&3\\
0&0&0&0&0&0&0&0&0&3&3&3&0&0&3&0&3&3\\
\hline
1&0&0&1&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
1&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&0\\
1&0&0&0&0&1&1&0&0&0&0&1&0&0&0&0&0&0\\
0&1&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\
0&1&0&0&1&0&1&0&0&1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&1&0&0&1&0&1&0&0&0&0&0&0&0\\
0&0&1&1&0&0&1&0&0&0&1&0&0&0&0&0&0&0\\
0&0&1&0&1&0&0&0&1&0&0&1&0&0&0&0&0&0\\
0&0&1&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0
\end{array}\right]
$$

Fig. 1. The incidence matrix of a 2-(21,6,1;{1,3}) point-sum design

The second studied design condition was introduced by Powlesland [6] and affects sets of points whose total weight is t. These designs, now termed *weight-sum designs*, will not be examined by this paper.

It is necessary to be able to differentiate between point-weight incidence structures which are truly different and those that merely have a linear difference in the weights of the points. In order to do this we define the notion of equivalence.

Definition 4 *Two point-weight incidence structures* $\mathcal{S}_1 = (V_1, \mathcal{B}_1, I_1, w_1)$ *and* $\mathcal{S}_2 = (V_2, \mathcal{B}_2, I_2, w_2)$ *are equivalent if there exists an isomorphism* $\theta : \mathcal{S}_1 \to \mathcal{S}_2$ *and a constant* $\mu \in \mathbb{Q}$ *such that:*

(1) θ *shows that* $(V_1, \mathcal{B}_1, I_1)$ *and* $(V_2, \mathcal{B}_2, I_2)$ *are isomorphic as designs,*
(2) and $w(x) = \mu \cdot w(\theta(x))$ *for all* $x \in V$.

Hence we have that every point-weight incidence structure is equivalent to a point-weight incidence structure for which $\gcd(W) = 1$. Therefore, for the remainder of this paper, we will assume that every point-weight incidence structure has $\gcd(W) = 1$.

3 Design Conditions on t Points

In this section we will examine point-weight designs with a design condition that assumes that every set of "identical" t points is incident with the same number of blocks. For the purposes of this section, two sets of points are identical if they contain the same number of points of each weight. Both point-sum designs and row-sum designs (which will be introduced in Sect. 4) are examples of point-weight designs that behave in this way. This behaviour can be formalised in the following way:

Definition 5 *Let $(V, \mathcal{B}, I, w)$ be a point-weight incidence structure and $f : \binom{V}{t} \to \mathbb{Z}^+$. We say that $(V, \mathcal{B}, I, w)$ is a $t-(v, k, f; W)$ design if*

(1) the sum of the weights of the points is v,
(2) the sum of the weights of the points on any block is k,
(3) if $S \subseteq V$ is a set of t points then S is incident with $f(S)$ blocks,
(4) if $S_1, S_2 \subseteq V$ are sets of t points that contain equally many points of each weight then $f(S_1) = f(S_2)$,
(5) the image of the weight function is W.

Point-sum designs form a particular class of $t-(v, k, f; W)$ design where f is the constant function $f(S) = \lambda$. The main effort in this section will be put toward proving Theorem 6.

Theorem 6 *Suppose t is a positive integer greater than 1 and $\mathcal{S} = (V, \mathcal{B}, I, w)$ is a $t-(v, k, f; W)$ design for some function f. Then, if $1 \leq s \leq t$ and S_1, $S_2 \subseteq V$ are sets of s points that contain equally many points of each weight, S_1 and S_2 are incident with the same number of blocks.*

In particular this means that any two points of the same weight are incident with the same number of blocks.

In order to prove Theorem 6 we will need to use the concept of a derived structure for a point-weight incidence structure. The definition is the natural extension of the definition for a classical design.

Definition 7 (Derived structure) *Suppose $\mathcal{S}$ is any $(V, \mathcal{B}, I, w)$ point-weight incidence structure and $T \subset V$ is a proper subset of the set of points. The derived structure of $\mathcal{S}$ at T, written $\mathcal{S}_T = (V_T, \mathcal{B}_T, I_T, w_T)$, is given by*

(1) $\mathcal{B}_T = \{B \in \mathcal{B} : T \subseteq B\}$,
(2) $V_T = V \setminus (T \cup \{y \in V : \forall B \in \mathcal{B}_T, y \notin B\})$,
(3) $I_T = I \cap (V_T \times \mathcal{B}_T)$,
(4) $w_T(x) = w(x)$ for all $x \in V_T$.

In other words, we obtain $\mathcal{S}_T$ from $\mathcal{S}$ by removing the points T, all blocks that do not contain T and all points that are no longer incident with any block. The weight function of $\mathcal{S}_T$ is the natural restriction of w to V_T.

Lemma 8 *Suppose $\mathcal{S}$ is a $t-(v,k,f;W)$ point-weight design. If T is a set of s points with $1 \leq s < t$ then $\mathcal{S}_T$ is a $(t-s)-(v-\sigma(T), k-\sigma(T), f'; W')$ point weight-design where*

$$f'(S) = f(S \cup T) \text{ for any set } S \text{ of } s \text{ points in } \mathcal{S}_T \tag{3.1}$$

and

$$W \setminus w(T) \subseteq W' \subseteq W\,. \tag{3.2}$$

We will proceed to prove Theorem 6 by induction, hence we begin showing that it holds for the special case when $t = 2$.

Lemma 9 *If $\mathcal{S}$ is a $2-(v,k,f;W)$ point-weight design then any two points of the same weight are incident with the same number of blocks.*

PROOF. Pick any $x \in V$ and let r_x be the number of blocks with which x is incident. The number of blocks in $\mathcal{S}_{\{x\}}$ is equal to r_x. Now consider counting the weighted flags of $\mathcal{S}_{\{x\}}$:

$$\sum_{(y,B)\in I_{\{x\}}} w(y) = \sum_{B\in\mathcal{B}_{\{x\}}} \sum_{y\in B} w(y) = r_x(k - w(x)) \tag{3.3}$$

However, we also have:

$$\sum_{(y,B)\in I_{\{x\}}} w(y) = \sum_{y\in V_{\{x\}}} \sum_{B\ni y} w(y) = \sum_{y\in V_{\{x\}}} f'(y)w(y) \tag{3.4}$$

where $f'(y) = f(\{x,y\})$. Hence,

$$r_x = \frac{1}{k - w(x)} \sum_{y\in V_{\{x\}}} f'(y)w(y) \tag{3.5}$$

and so $r_x = r_y$ whenever $w(x) = w(y)$. $\square$

We are now in a position to prove Theorem 6 using an idea of Riordan [7].

PROOF OF THEOREM 6. We use induction on t and note that the special case where $t = 2$ is proven in Lemma 9.

Let $t \geq 3$ and suppose, as an induction hypothesis, that every $s-(t,k,f;W)$ point-weight design with $s<t$ has the property that if S_1 and S_2 are sets of points in that design such that

(1) S_1 and S_2 contain equally many points of each weight, and
(2) $1 \leq |S_1| = |S_2| \leq s$

then S_1 and S_2 are incident with the same number of blocks (i.e. $\iota(S_1) = \iota(S_2)$).

Now consider a point-weight incidence structure $\mathcal{S} = (V, \mathcal{B}, I, w)$ that is a $t-(v,k,f;W)$ point-weight design. Let T be a set of of $t-s$ points where $1 \leq s < t$. We will attempt to find an expression for the number of blocks that are incident with the set T, r_T say. We note that r_T is equal to the number of blocks of $\mathcal{S}_T$ and that $\mathcal{S}_T$ is a $s-(v-\sigma(T), k-\sigma(T), f'; W')$ point-weight design where $f'(S) = f(S \cup T)$ for any set S of s points in $\mathcal{S}_T$.

Let
$$I' = \{(S,B) : S \subseteq V_T,\ B \in \mathcal{B}_T,\ |S| = s \text{ and } S \subseteq B\} \tag{3.6}$$
and consider the sum
$$\sum_{(S,B)\in I'} \left(\prod_{z\in S} w(z) \right). \tag{3.7}$$
This sum can be evaluated in two ways. The most obvious evaluation is as follows.

$$\begin{aligned} \sum_{(S,B)\in I'} \left(\prod_{z\in S} w(z) \right) &= \sum_{S\subseteq V_T:|S|=s} \left\{ \sum_{B\in\mathcal{B}_T:S\subseteq B} \left(\prod_{z\in S} w(z) \right) \right\} \\ &= \sum_{S\subseteq V_T:|S|=s} \left\{ f'(S) \prod_{z\in S} w(z) \right\} \end{aligned} \tag{3.8}$$

Note that this sum depends not on the specific set T but only on the number of points of each weight contained in T.

The second method for evaluating the sum is more complicated. Consider the classical incidence structure $\mathcal{S}_T^* = (V_T^*, \mathcal{B}_T^*, I_T^*)$ given by:

$$V_T^* = \{x_i : x \in V_T \text{ and } 1 \leq i \leq w(x)\}, \tag{3.9}$$
$$\mathcal{B}_T^* = \mathcal{B}_T, \tag{3.10}$$
$$I_T^* = \{(x_i, B) : (x,B) \in I\}. \tag{3.11}$$

In other words, $\mathcal{S}_T^*$ is formed by changing every point x of S_T into $w(x)$ points and extending incidence in the natural way.

Now given a set S of s points in $\mathcal{S}_T$ there exist $\prod_{z\in S} w(z)$ ways of choosing a set S^* of s points in $\mathcal{S}_T^*$ such that each point of S^* was obtained from a distinct point of S. Furthermore there exist $r_T\binom{k-\sigma(T)}{s}$ ways of picking a pair (S^*, B) where S^* is a set of s points of $\mathcal{S}_T^*$ and $B \in \mathcal{B}_T$ is a block that contains S^*. However this includes the sets S^* whose members are obtained from a set of points S in $\mathcal{S}_T$ of size less than s. We shall attempt to calculate how many such "bad" sets exist.

Suppose S is a set of $n < s$ points of $\mathcal{S}_T$ and label these points $z^{(1)}, \ldots, z^{(n)}$. Let

$$J_n = \{(j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n : \sum_{i=1}^{n} j_i = s \text{ and } j_i > 0 \text{ for all } 1 \leq i \leq n\} \quad (3.12)$$

The number of ways of choosing an ordered pair (S^*, B) where S^* is a set of s points of $\mathcal{S}_T^*$ obtained from a set S of $n < s$ points of $\mathcal{S}_T$ and B is a block in $\mathcal{S}_T$ that contains S is:

$$p(S) = \iota_T(S) \sum_{(j_1,\ldots,j_n)\in J_n} \prod_{i=1}^{n} \binom{w(z^{(i)})}{j_i}, \quad (3.13)$$

where $\iota_T(S)$ is the number of blocks that S is incident with in $\mathcal{S}_T$. Note that, by the induction hypothesis, this sum depends not on the specific set T but only on the number of points of each weight contained in T.

Hence,

$$\sum_{(S,B)\in I'} \left(\prod_{z\in S} w(z) \right) = r_T \binom{k-\sigma(T)}{s} - \sum_{i=1}^{s-1} \left\{ \sum_{S\subseteq V_T:|S|=i} p(S) \right\}. \quad (3.14)$$

Equating this equation with Eq. 3.8 gives an expression for r_T that does not depend upon the specific set T but only upon the number of points of each weight contained in T. Hence any two sets of s points that contain the same number of points of each weight are incident with the same number of blocks. □

4 Row-Sum Designs

We now present a new class of point-weight designs, called *row-sum point-weight designs* or *row-sum designs* when the context is clear.

Definition 10 (Row-sum point-weight design) *A $(V, \mathcal{B}, I, w)$ point-weight incidence structure is a $\pi_t - (v, k, \lambda; W)$ point-weight design if*

(1) the sum of the weights of the points is v.
(2) the sum of the weights of the points on any one block is k.

(3) any set S of t points is incident with

$$\iota(S) = \frac{\lambda}{\prod_{x \in S} w(x)}$$

points.

(4) the image of the weight function is W.

Notice that this class is an example of a $t-(v,k,f;W)$ design with

$$f(S) = \frac{\lambda}{\prod_{x \in S} w(x)}$$

for all sets S of t points. This somewhat arbitrary choice of design condition is interesting because it allows us to extend the theory of incidence matrices to point-weight designs in a very natural way. A (weighted) incidence matrix for a point-weight incidence structure was first defined in [4].

Definition 11 (Incidence matrix) *Suppose $\mathcal{S} = (V, \mathcal{B}, I, w)$ is a point-weight incidence structure. M is an incidence matrix for $\mathcal{S}$ if there exists an enumeration of the point set $V = \{x_1, x_2, \ldots, x_u\}$ and an enumeration of the block set $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ such that*

$$M_{i,j} = \begin{cases} w(x_i) & \text{if } (x_i, B_j) \in I \\ 0 & \text{otherwise} \end{cases}$$

Many of the elementary theorems concerning classical incidence matrices also apply here. In particular the following theorem holds for (and, indeed, is a motivating factor for the definition of) row-sum point-weight designs.

Lemma 12 *Suppose M is an incidence matrix for the $\pi_2-(v,k,\lambda;W)$ point-weight design with u points then*

$$MM^T = diag(w(x_1)^2 r_{x_1}, \ldots, w(x_u)^2 r_{x_u}) + \lambda(J - I)$$

where J is the matrix with every entry equal to 1 and r_{x_i} is the number of blocks with which the point x_i is incident ($1 \le i \le u$). Furthermore

$$det(MM^T) = \prod_{x \in V} (w(x)^2 r_x - \lambda) + \lambda \sum_{x \in V} \prod_{y \in V \setminus \{x\}} (w(y)^2 r_y - \lambda) \tag{4.1}$$

$$= (1 + \lambda \sum_{x \in V} \frac{1}{w(x)^2 r_x - \lambda}) \prod_{x \in V} (w(x)^2 r_x - \lambda) \tag{4.2}$$

Row-sum designs are possibly most interesting from a combinatorial point of view, however they can also be viewed as a type of n-ary block design [3, 10].

For our purposes, a n-ary block design is a block design in which the incidence matrix only contains entries from the set $\{0, 1, \dots, n-1\}$.[1] Furthermore, a block design is said to be *proper* if each block contains the same number of treatments and *pairwise balanced* if

$$MM^T = D + \lambda J\,,$$

where D is a diagonal matrix and M in the incidence matrix of the design. Hence, row-sum point-weight designs can be thought of as proper, pairwise balanced n-ary block designs (where $n-1$ is the largest integer in W).

Row-sum designs share several properties with classical designs.

Lemma 13 (Fisher's Inequality) *If $\mathcal{S}$ is a $\pi_2 - (v, k, \lambda; W)$ point-weight design with u points and b blocks, and there exist m points x which satisfy $w(x)^2 r_x \leq \lambda$ then $b \geq u - m$.*

PROOF. Let M be an incidence matrix for $\mathcal{S}$ and suppose that M' is the incidence matrix formed by removing the rows of M which correspond to the points of $\mathcal{S}$ that satisfy $w(x)^2 r_x \leq \lambda$. We still have that

$$M'M'^T = diag(w(x_1)^2 r_{x_1}, \dots, w(x_{u-m})^2 r_{x_{u-m}}) + \lambda(J - I)$$

for suitably sized matrices I and J, and for a suitable labelling of the points of V. This means that $det(M'M'^T) \neq 0$ as $w(x)^2 r_x - \lambda > 0$ for all appropriate x. Hence we know that

$$b \geq rank(M) \geq rank(MM^T) \geq rank(M'M'^T) = u - m \qquad (4.3)$$

which gives us the result. $\square$

It is clear that the terms $w(x)^2 r_x - \lambda$ play an important role in determining the structure of a row-sum design. In a classical 2-design, the equivalent term would be $r - \lambda$ where r is the (constant) number of blocks with which any single point is incident. This is known as the order of the design. In a classical design, the order is always positive[2], whereas it is possible that $w(x)^2 r_x - \lambda \leq 0$ for some point x in a row-sum design.

[1] Some definitions of n-ary block design, including [8], insist that the incidence matrix must contain each value at least once, i.e. the incidence matrix contains n distinct integers. We only require that the entries in the incidence matrix are a subset of $\{0, 1, \dots, n-1\}$.

[2] Technically, this is only true designs for which the block size is less than the total number of points.

Definition 14 *Let $(V, \mathcal{B}, I, w)$ be a point-weight incidence structure. We define the order of a point $x \in V$ to be $n_x = w(x)^2 r_x - \lambda$.*

A point is said to be awkward *if it has zero order and* difficult *if it has negative order. A point-weight incidence structure is said to be awkward (resp. difficult) if it contains a point which is awkward (resp. difficult).*

The structure of an awkward or difficult design is constrained by the following lemmas.

Lemma 15 *Suppose x is an awkward point in a $\pi_2 - (v, k, \lambda; W)$ point-weight design. Then*

(1) $w(x) \leq w(y)$ for all $y \in V$,
(2) and if $w(x) = w(z)$ for some point z then x and z are incident with exactly the same blocks. Hence z is an awkward point too.

PROOF. Let x be an awkward point. Hence if x is incident with r_x blocks then $w(x)^2 r_x = \lambda$. If $y \neq x$ is another point then

$$\frac{\lambda}{w(x)w(y)} = \iota(x, y) \leq r_x = \frac{\lambda}{w(x)^2}$$

and so $w(x) \leq w(y)$.

Now suppose z is another point of minimal weight and suppose z is incident with r_z blocks. Since $w(x) = w(z)$ we have that $r_z = \frac{\lambda}{w(x)^2} = \frac{\lambda}{w(x)w(z)}$ by Theorem 6. Thus $r_z = \iota(x, z)$ and every block that contains x also contains z. Hence x and z are incident with exactly the same blocks and z is awkward. $\square$

Therefore if a design is awkward then all the points of minimal weight are awkward and incident with exactly the same blocks. An example of an awkward row-sum design is given in Figure 2.

By a similar method we can show the following.

Lemma 16 *If x is a difficult point of a $\pi_2 - (v, k, \lambda; W)$ point-weight design then $w(x) < w(y)$ for all $y \in V \setminus \{x\}$.*

Corollary 17 *There exists no $\pi_2 - (v, k, \lambda; W)$ point-weight designs that are both awkward and difficult.*

An example of a difficult row-sum design is given in Figure 3.

$$\left[\begin{array}{ccccccccc|cc}
1&1&1&1&1&1&1&1&1&0&0\\
1&1&1&1&1&1&1&1&1&0&0\\
1&1&1&1&1&1&1&1&1&0&0\\
\hline
3&0&0&0&0&3&3&0&0&3&0\\
3&3&0&0&0&0&0&3&0&0&3\\
0&3&3&0&0&0&0&0&3&3&0\\
0&0&3&3&0&0&3&0&0&0&3\\
0&0&0&3&3&0&0&3&0&3&0\\
0&0&0&0&3&3&0&0&3&0&3
\end{array}\right]$$

Fig. 2. The incidence matrix of a $\pi_2 - (21, 9, 9; \{1, 3\})$ awkward design.

$$\left[\begin{array}{cccc}
2&2&2&2\\
\hline
3&3&3&0\\
3&3&0&3\\
3&0&3&3\\
0&3&3&3
\end{array}\right]$$

Fig. 3. The incidence matrix of a $\pi_2 - (14, 11, 18; \{2, 3\})$ difficult design.

We finish this section, and bring ourselves back full circle, by recalling that a row-sum point-weight design is an example of a point-weight design with a design condition on t points. Hence we can apply Lemma 9 and Theorem 6 and obtain the following useful results.

Lemma 18 *If $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ design with u points and T is a set of $t-1$ points of $\mathcal{S}$ then T is incident with r_T blocks where*

$$r_T = \frac{\lambda(u - t + 1)}{(k - \sigma(T))} \prod_{x \in T} \frac{1}{w(x)} . \tag{4.4}$$

This leads to several nice corollaries.

Corollary 19 *If $\mathcal{S}$ is a $\pi_2 - (v, k, \lambda; W)$ design with u points and x is a point of $\mathcal{S}$ then x is incident with r_x blocks where*

$$r_x = \frac{\lambda(u - 1)}{(k - w(x))w(x)} . \tag{4.5}$$

Corollary 20 *If $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ design with $|W| \geq 2$ and $t > 1$ then*

$\mathcal{S}$ is not $\pi_{t-1} - (v, k, \lambda'; W)$ for any $\lambda' \in \mathbb{Z}^+$.

This last corollary mirrors the following result of Horne [4].

Lemma 21 *If $\mathcal{S}$ is a $t - (v, k, \lambda; W)$ design with $|W| \geq 2$, $t > 1$ and $v > k$ then $\mathcal{S}$ is not a $(t-1) - (v, k, \lambda'; W)$ for any $\lambda' \in \mathbb{Z}^+$.*

5 Point-Sum and Row-Sum Designs

Since there are now two types of design condition that specify the number of blocks a set of t points lies upon it is natural to ask if a point-weight incidence structure could ever be both a point-sum and a row-sum point-weight design. It is easy to show that, essentially, there are no non-trivial structures that are simultaneously row-sum and point-sum designs.

Lemma 22 *If a point-weight incidence structure $\mathcal{S}$ is both a $\pi_t - (v, k, \lambda; W)$ and a $t - (v, k, \lambda'; W)$ point-weight design, and $\mathcal{S}$ contains more than t points, then $|W| = 1$.*

We now investigate the possibility that a point-weight incidence structure can be both a row-sum and a point-sum design for different values of t. In order to do this we recall a lemma from [4].

Lemma 23 *If $\mathcal{S}$ is a $2-(v, k, \lambda; W)$ design and x is a point then x is incident with*

$$r_x = \lambda \frac{v - w(x)}{k - w(x)} \tag{5.1}$$

blocks.

Note that this can also be thought of as a corollary of Lemma 9 and Theorem 6. Now we are in a position to prove that a point-weight incidence structure cannot be both a $\pi_t - (v, k, \lambda; W)$ design and a $(t+1) - (v, k, \lambda'; W)$ design.

Theorem 24 *If a point-weight incidence structure $\mathcal{S}$ is both a $\pi_t - (v, k, \lambda; W)$ and a $(t+1) - (v, k, \lambda'; W)$ point-weight design, and $\mathcal{S}$ contains at least $t+1$ points, then $|W| = 1$.*

PROOF. We will use induction on t.

Consider the case when $t = 1$, i.e. $\mathcal{S}$ is both a $\pi_1 - (v, k, \lambda; W)$ and a $2 - (v, k, \lambda'; W)$ point-weight design, and suppose that $|W| > 1$. Let x be a point of $\mathcal{S}$. If x is incident with r_x blocks then

$$r_x = \frac{\lambda}{w(x)} = \lambda' \frac{v - w(x)}{k - w(x)} \tag{5.2}$$

Hence, for all points x in $\mathcal{S}$ we have that $w(x)$ is a solution of the equation:

$$X^2 - \frac{\lambda + \lambda' v}{\lambda'} X + \frac{\lambda k}{\lambda'} = 0\,. \tag{5.3}$$

Therefore $|W| = 2$.

Let x and y be two points of different weights. It is clear that

$$X^2 - \frac{\lambda + \lambda' v}{\lambda'} X + \frac{\lambda k}{\lambda'} = (X - w(x))(X - w(y)) \tag{5.4}$$

and so

$$v \geq w(x) + w(y) = v + \frac{\lambda}{\lambda'} \tag{5.5}$$

which is a contradiction. Hence we must have that $|W| = 1$.

Let $t > 1$. Suppose, as an induction hypothesis, that any point-weight incidence structure $\mathcal{S}$ that is both a $\pi_s - (v, k, \lambda; W)$ and a $(s+1) - (v, k, \lambda'; W)$ point-weight design with $1 \leq s < t$ (and at least $s + 1$ points) has $|W| = 1$.

Now suppose that $\mathcal{S}$ is a point-weight incidence structure that is both a $\pi_t - (v, k, \lambda; W)$ and a $(t+1) - (v, k, \lambda'; W)$ point-weight design with at least $t+1$ points and $|W| \geq 2$. If x is any point of $\mathcal{S}$ then, by Lemma 8, we have that the derived structure of $\mathcal{S}$ at x, $\mathcal{S}_x$, is both a

(1) $\pi_{t-1} - (v - w(x), k - w(x), \frac{\lambda}{w(x)}; W')$ point-weight design, and a
(2) $t - (v - w(x), k - w(x), \lambda'; W')$ point-weight design

where $W \setminus \{w(x)\} \subseteq W' \subseteq W$. We will divide the problem into two cases: those cases in which there exists only one point of each weight and those cases in which there exists more than one point of some weight.

If there exist two points, x and x', of the same weight then the derived structure of $\mathcal{S}$ at x has $W' = W$ and so $W' \geq 2$. This is a contradiction to the induction hypothesis.

If no two points have the same weight then $W' = W \setminus \{w(x)\}$ and, by the induction hypothesis, $|W'| = 1$. Hence $\mathcal{S}$ contains only two points, which is a contradiction to the fact that $\mathcal{S}$ contains at least $t + 1$ points and $t > 1$.

Hence there exist no point-weight incidence structures that contain at least $t+1$ points and are both $\pi_t - (v, k, \lambda; W)$ and $(t+1) - (v, k, \lambda'; W)$ point-weight designs. $\square$

Along the lines of the above theorem, we make the following conjecture.

Conjecture 25 *If a point-weight incidence structure* $\mathcal{S}$ *is both a* $\pi_s - (v, k, \lambda; W)$ *and a* $t - (v, k, \lambda'; W)$ *point-weight design where* $\mathcal{S}$ *contains at least* t *points and* $s < t$ *then* $|W| = 1$*.*

This could be proved by showing that any $t - (v, k, \lambda'; W)$ design was not also a $\pi_1 - (v, k, \lambda; W)$ design. The results of Theorem 6 show that, for a $t - (v, k, \lambda'; W)$ design, the number of blocks a single point is incident with is the same for all points of the same weight. This would be consistent with the incidence structure being a $\pi_1 - (v, k, \lambda; W)$ design. However the formula given in Theorem 6 for the number of blocks with which a single point x is incident in a $t - (v, k, \lambda'; W)$ design is highly complex, even for $t = 3$. In contrast, the formula for the number of blocks with which a point x is incident in a $\pi_1 - (v, k, \lambda; W)$ design is very simple. It seems unlikely that the two formulas can be reconciled.

A proof of this conjecture would allow us to put a lower bound on the value of t for which a row-sum design could also also be a point-sum design. As we shall see in the next section, this conjecture cannot be extended to postulate that there exists no point-weight incidence structures with $|W| > 1$ that are both point-sum and row-sum point-weight designs.

6 Constructing Row-Sum Designs

In this section we counter the negative results of the last section by giving a general method for constructing structures that are simultaneously $t - (v, k, \lambda'; W)$ designs and $\pi_{t+1} - (v, k, \lambda; W)$ designs. The proof that the following construction is valid is left to the reader.

Lemma 26 *Let* λ', λ, t *be positive integers and suppose that* B *is the incidence matrix of a classical* $(t + 1) - (\lambda' + t - 1, \lambda + t, \lambda)$ *design. The point-weight incidence structure defined by the incidence matrix*

$$\left[\begin{array}{c|c} \lambda \dots \lambda & 0 \dots 0 \\ \hline & \\ A & B \\ & \end{array}\right]$$

where A *is the* $(\lambda' + t - 1) \times \binom{\lambda'+t-1}{t}$ *matrix consisting of all the possible 0,1-column vectors containing exactly* t *ones, is both*

$$\left[\begin{array}{cccccccccc|ccccc} 2&2&2&2&2&2&2&2&2&2&0&0&0&0&0 \\ \hline 1&1&1&1&0&0&0&0&0&0&0&1&1&1&1 \\ 1&0&0&0&1&1&1&0&0&0&1&0&1&1&1 \\ 0&1&0&0&1&0&0&1&1&0&1&1&0&1&1 \\ 0&0&1&0&0&1&0&1&0&1&1&1&1&0&1 \\ 0&0&0&1&0&0&1&0&1&1&1&1&1&1&0 \end{array}\right]$$

Fig. 4. The incidence matrix of a point-weight incidence structure that is both a $\pi_3 - (6, 4, 2; \{1, 2\})$ and a $2 - (6, 4, 4; \{1, 2\})$ point-weight design.

(1) a $\pi_{t+1} - (\lambda + \lambda' + t - 1, \lambda + t, \lambda; \{1, \lambda\})$ row-sum design, and
(2) a $t - (\lambda + \lambda' + t - 1, \lambda + t, \lambda'; \{1, \lambda\})$ point-sum design.

We give an example of a point-weight incidence structure that is both a $\pi_3 - (6, 4, 2; \{1, 2\})$ and a $2 - (6, 4, 4; \{1, 2\})$ in Fig 4. Furthermore we show that this method of construction is the only way of constructing a point-weight incidence that is both a $\pi_{t+1} - (v, k, \lambda; W)$ and a $t - (v, k, \lambda'; W)$ design.

Theorem 27 *If $\mathcal{S}$ is a point-weight incidence structure that is both a $\pi_{t+1} - (v, k, \lambda; W)$ and a $t - (v, k, \lambda'; W)$ design, contains more than $t + 1$ points and for which $|W| \geq 2$ then $\mathcal{S}$ has an incidence matrix of the form*

$$\left[\begin{array}{c|c} \lambda \ldots \lambda & 0 \ldots 0 \\ \hline & \\ A & B \\ & \end{array}\right]$$

where A is $(\lambda' + t - 1) \times \binom{\lambda'+t-1}{t}$ matrix consisting of all the possible 0,1-column vectors containing exactly t ones and B is the incidence matrix of a $(t + 1) - (\lambda' + t - 1, \lambda + t, \lambda)$ classical design.

PROOF. Again we use induction on t.

Suppose that $t = 1$, i.e. $\mathcal{S}$ is a $\pi_2 - (v, k, \lambda; W)$ and a $1 - (v, k, \lambda'; W)$ design that contains $u > 2$ points. So, for any point x of $\mathcal{S}$ we have that x is incident with λ' blocks. Hence, by Lemma 19, we have that

$$\frac{\lambda(u - 1)}{(k - w(x))w(x)} = \lambda \tag{6.1}$$

which means that $|W| = 2$. This means that $W = \{w(x), w(y)\}$ for some points x and y. Without loss of generality assume that $w(x) < w(y)$. So, from equation 6.1, we have that

$$(k - w(x))w(x) = (k - w(y))w(y)) \tag{6.2}$$

and so

$$(k - w(x) - w(y))(w(x) - w(y)) = 0\,. \tag{6.3}$$

Hence $k = w(x) + w(y)$. Therefore there exists only one point y of weight $w(y)$ and, since $u \geq 3$, there are multiple points of weight $w(x)$. Since there exists at most one block that contains both x and y we must have that $\lambda = w(x)w(y)$. Since there exists multiple points of weight $w(x)$ we must also have that $w(x)$ divides $w(y)$ and so $w(x) = 1$ by the co-primality of W. Therefore the incidence matrix for $\mathcal{S}$ must be of the form

$$\left[\begin{array}{c|c} \lambda \ldots \lambda & 0 \ldots 0 \\ \hline & \\ I & B \\ & \end{array}\right]$$

where B is a 0,1-matrix. It is obvious from the incidence matrix that B must be a $2 - (u-1, \lambda+1, \lambda')$ classical design. Hence the theorem holds for $t = 1$.

Let $t > 1$. Suppose as an induction hypothesis, that the theorem holds for all point-weight incidence structures that both a $\pi_{s+1} - (v, k, \lambda; W)$ and a $s - (v, k, \lambda'; W)$ design with $|W| \geq 2$ and more than $s + 1$ points where $1 \leq s < t$. Consider a point-weight incidence structure $\mathcal{S}$ that is both a $\pi_{t+1} - (v, k, \lambda; W)$ and a $t - (v, k, \lambda'; W)$ design and let u be the number of points in $\mathcal{S}$. Note that, for any point x in $\mathcal{S}$, we have that the derived structure of $\mathcal{S}$ at x is both a

(1) $\pi_t - (v - w(x), k - w(x), \frac{\lambda}{w(x)}; W')$ design, and a
(2) $(t-1) - (v - w(x), k - w(x), \lambda'; W')$ design

where $W \setminus \{w(x)\} \subseteq W' \subseteq W$, and $\mathcal{S}$ contains more than t points.

Suppose that $\mathcal{S}$ contains no two points that have the same weight. For any point x in $\mathcal{S}$ we have that $\mathcal{S}_x$ has a weight set $W' = W \setminus \{w(x)\}$. By the induction hypothesis $|W'| \leq 2$, which is a contradiction to the fact that $\mathcal{S}$ has more than $t + 1$ points. Hence there must exist two points x_1, x_2 which have the same weight.

We therefore have that $\mathcal{S}_{x_1}$ fulfils all the criteria in the induction hypothesis and so $\mathcal{S}_{x_1}$ has an incidence matrix of the form

$$\left[\begin{array}{c|c} \lambda \dots \lambda & 0 \dots 0 \\ \hline & \\ A' & C' \\ & \end{array}\right]$$

where A' is the $(\lambda' + t - 2) \times \binom{\lambda'+t-2}{t-1}$ matrix consisting of all the 0,1-column vectors that contain exactly $t-1$ ones, and C' is the incidence matrix of a $t-(\lambda'+t-2, \lambda+t-1, \lambda)$ classical design. In particular this means that $w(x_2) = 1$. Hence the incidence matrix of $\mathcal{S}$ must be of the form

$$\left[\begin{array}{c|c|c|c} \lambda \dots \lambda & \lambda \dots \lambda & 0 \dots 0 & 0 \dots 0 \\ \hline 1 \dots 1 & 0 \dots 0 & 1 \dots 1 & 0 \dots 0 \\ \hline & & & \\ A' & B' & C' & D' \\ & & & \end{array}\right]$$

Hence $v = \lambda + \lambda' + t - 1$ and $k = \lambda + t$.

Let z be the point of weight λ. The point-weight incidence structure $\mathcal{S}_z$ is a $t-(u-1, t, 1)$ classical design. This means that the incidence matrix of $\mathcal{S}_z$ must be a $(\lambda' + t - 1) \times \binom{\lambda'+t-1}{t}$ matrix consisting of all the possible 0,1-column vectors that contain exactly t ones. Similarly it is not hard to see that the structure defined by the incidence matrix

$$\left[\begin{array}{c|c} 1 \dots 1 & 0 \dots 0 \\ \hline & \\ C' & D' \\ & \end{array}\right]$$

must be a $(t+1)-(u-1, \lambda+t, \lambda)$ classical design, as $\mathcal{S}$ is a $(t+1)-(v, k, \lambda; W)$ point-weight design. Hence the theorem holds. □

We are forced to include the condition that $\mathcal{S}$ must have more than $t+1$ points to exclude the trivial design which consists of a single block containing all the points. The combination of Lemma 26 and Theorem 27 allow us to make some simple statements about the parameters of these incidence structures.

Corollary 28 *If $\mathcal{S}$ is a point-weight incidence structure that is both a $\pi_{t+1} - (v, k, \lambda; W)$ and a $t - (v, k, \lambda'; W)$ design, contains more than $t+1$ points and for which $|W| \geq 2$ then $v = \lambda + \lambda' + t - 1$, $k = \lambda + t$ and $W = \{1, \lambda\}$.*

In a manner similar to Section 5, we provide the following conjecture.

Conjecture 29 *For every $1 \leq s \leq t$ there exists a point-weight incidence structure that is both a $\pi_t - (v, k, \lambda; W)$ and a $s - (v, k, \lambda'; W)$ point-weight design and has both $|W| \geq 2$ and more than t points.*

Again, the main obstacle to providing a proof for this conjecture is the complexity of the formula given in Theorem 6. Further construction techniques for row-sum point-weight designs can be found in [2].

7 Conclusions and Open Problems

This paper has examined the combinatorial conditions imposed on point-weight designs that have a design conditions that mandate the number of blocks with which a set of t points is incident. It has introduced a new class of designs, row-sum point-weight designs, that have this property and thoroughly examined the relationship between these new design and the already established class of point-sum point-weight designs.

It has become apparent that the relationship between the values of t for which a point-weight incidence structure is a design with a design conditions on t points is very interesting. For a given incidence structure $\mathcal{S}$, consider marking a copy of the integers with spots depending whether $\mathcal{S}$ is a row-sum or point-sum design with a design condition on t points. Mark a value $t \in \mathbb{Z}^+$ green (resp. blue) if there exists a constant λ such that $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ design (resp. a $t - (v, k, \lambda; W)$ design). In this case we have shown that all blue spots (and green spots) are at least 2 away from each other, and we have conjectured that all the blue spots are below all the green spots.

This paper leaves us with many interesting open problems:

(1) Is it possible to find a proof or counter-example to Conjecture 25? That is, is it possible to show that, for any $1 \leq s \leq t$, there exists no point-weight incidence structure that is both a $\pi_s - (v, k, \lambda; W)$ and a $t - (v, k, \lambda; W)$ design and has both $|W| \geq 2$ and more than t points?

(2) Is it possible to find a proof or counter-example to Conjecture 29? That is, is it possible to show that, for any $1 \leq s \leq t$, there exists a point-weight incidence structure that is both a $\pi_t - (v, k, \lambda; W)$ design and a $s - (v, k, \lambda'; W)$ design with $|W| \geq 2$ and more than t points?

(3) Indeed, it is not yet clear whether a point-weight incidence structure could be a row-sum or a point-sum point-weight design for two different values of t (see Lemma 20 and Lemma 21). For a point-weight incidence structure $\mathcal{S}$, how many values of t are there for which $\mathcal{S}$ is a point-sum (or row-sum) design?

(4) All the examples in this paper, indeed all the known examples, of $\pi_t - (v, k, \lambda; W)$ designs with more than t points and $|W| > 1$ have $|W| = 2$. Dare we conjecture that there exists no row-sum point-weight designs with a weight set of size three or greater?

8 Acknowledgements

I would like to thank both Prof. Peter Wild and Prof. Fred Piper for both introducing me to this area and for many useful discussions. I would also like to thank Dr. Oliver Riordan for taking a moment to point me in the right direction.

References

[1] T. Beth, D. Jungnickel, and H. Lenz. *Design Theory*, volume 1. Cambridge University Press, second edition, 1999.

[2] A. W. Dent. *On the Theory of Point-Weight Designs.* PhD thesis, Royal Holloway, University of London, 2001.

[3] A. Hedayat and W. Federer. Pairwise and variance balanced incomplete block designs. *Annals of the Institue of Statistical Mathematics*, 26:331–338, 1974.

[4] R. Horne. *On Point-Weighted Designs.* PhD thesis, Royal Holloway, University of London, 1996.

[5] D. R. Hughes and F. Piper. *Design Theory.* Cambridge University Press, 1985.

[6] T. Powlesland. *Designs With Weight Assignment.* PhD thesis, Royal Holloway, University of London, 1995.

[7] O. Riordan. Personal Communication.

[8] M. Shafiq and W. Federer. Generalized n-ary balanced block designs. *Biometrika*, 66:115–123, 1979.

[9] A. Street and D. Street. *Combinatorics of Experimental Designs.* Oxford Univeristy Press, 1987.

[10] K. Tocher. The design and analysis of block experiments. *Journal of the Royal Statistical Society (Series B)*, 14:45–100, 1952.